Ending the Search for Component VaR

by
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Introduction

Present methods of calculating Value at Risk (VaR) prescribe two basic calculations: (a) the total, diversified VaR for a portfolio, or (b) the undiversified VaR for some portfolio subset. The portfolio subset might be all trades of a certain type or involving a certain asset, the individual trades themselves, or even the solitary (mapped) cashflows. Except in rare and exceptional cases, the undiversified VaRs of the components comprising a portfolio almost never aggregate to the diversified VaR of the portfolio. Neither do the undiversified VaRs provide any hint as to whether the corresponding components act to "hedge" the remainder of the portfolio, or instead serve only to increase its risk. This leads us to the search for a useful definition of **Component VaR**. A good definition of "Component VaR" has at least three desirable properties: (1) if the components partition the portfolio (i.e. are disjoint and exhaustive), then the Component VaRs should sum to the (diversified) portfolio VaR; (2) if the component were to be deleted from the portfolio, the Component VaR should tell us, at least approximately, how the portfolio VaR will change; and therefore, (3) Component VaR will be *negative* for components which act to hedge the remainder of the portfolio.

In this paper, we show that such a definition of Component VaR may be based upon the VaR-Delta concept (also known as "Del-VaR") introduced in an earlier article in this journal (RISK, May 1996, vol. 9, no. 5, pp.61-63). VaR-Delta is a portfolio metric appropriate to the analytic ("variance-covariance") methodology of VaR. The relationship of the VaR-Delta to the VaR is very analogous to the relationship between the option delta and the option price. In this case, however, it measures the sensitivity of VaR to the injection of a unit of cashflow in each dimension (or "vertex," per the J. P. Morgan RiskMetrics approach[1]) of the cashflow space. In the previous article also, it is shown how a new, candidate trade can be analyzed for its effect upon portfolio VaR. Perhaps surprisingly, the same technique can be applied to *existing* trades within a portfolio, to form a useful and meaningful definition of Component VaR.

Background

Let a portfolio of trades *P* be defined, and let *m(P)* be a cashflow "mapping" function, that is, a function which determines the amounts of cashflows on a set of defined vertices, given a portfolio. Let the vector **p** be the amounts of such vertex cashflows, i.e. **p =** *m(P),* stated in present-value, numeraire-based terms. Let **Q** be the horizon- and confidence level-scaled covariance matrix describing the covariance structure of the vertices. (For example, suppose a 5%, 1-day VaR is desired. Then the scale factor would be 1.645 x 1.645 / 262 = .0103, where 1.645 is the number of standard deviations of a 5% cumulative normal distribution, and 262 is the approximate number of trading days per year. This number would then multiply the covariance matrix describing the (annualized) vertex returns.) Then the portfolio VaR is given as:

$$VaR(\mathbf{p}) \equiv \sqrt{\mathbf{p}'\mathbf{Q}\mathbf{p}}$$

In the article previously cited, the VaR-Delta quantity was also calculated, being:

$$VaRDelta(\mathbf{p}) \equiv \nabla\sqrt{\mathbf{p}'\mathbf{Q}\mathbf{p}} = \frac{\mathbf{Q}\mathbf{p}}{\sqrt{\mathbf{p}'\mathbf{Q}\mathbf{p}}} = \frac{\mathbf{Q}\mathbf{p}}{VaR(\mathbf{p})}$$

Finally, it was also reported that the incremental VaR of a new, candidate trade *A* with cashflow map **a** = *m(A)* is (approximately) given by the inner product of VaR-Delta and *m(A)*, that is,

$$Incr(\mathbf{a},\mathbf{p}) = \mathbf{a}' \bullet VaRDelta(\mathbf{p}).$$

As pointed out in the previous paper, this formula can therefore be used to analyze the impact of the next trade upon portfolio VaR, possibly in a real-time setting, since this inner product is very quickly computed.

We next apply similar reasoning to a trade *presently extant* within the portfolio.

Development

The foundation of Component VaR rests upon a simple yet rather surprising theorem, namely that, for any cashflow map **p**, we have:

$$\mathbf{p}' \bullet VaRDelta(\mathbf{p}) = \mathbf{p}' \bullet \frac{\mathbf{Q}\mathbf{p}}{\sqrt{\mathbf{p}'\mathbf{Q}\mathbf{p}}} = \frac{\mathbf{p}'\mathbf{Q}\mathbf{p}}{\sqrt{\mathbf{p}'\mathbf{Q}\mathbf{p}}} = VaR(\mathbf{p}) \quad \text{[Additivity]}$$

In other words, the inner product of any cashflow map and its corresponding VaR-Delta is equal to its VaR. This remarkable theoretical relationship between the VaR function and its gradient (which flows from its homogeneity properties) provides the underpinning of the additivity properties supporting our definition of Component VaR.[2]

Suppose now that the cashflow map **p** derives from the arbitrary addition of component cashflow maps, namely,

$$\mathbf{p} = \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{p_i}$$

where N is the number of components. Then it follows from equation [Additivity] that

$$VaR(\mathbf{p}) = \mathbf{p}' \bullet VaRDelta(\mathbf{p}) = \sum_{\mathbf{i=1}}^{\mathbf{N}} \{\mathbf{p}_\mathrm{i}' \bullet VaRDelta(\mathbf{p})\}$$

Defining the component VaRs as being those terms on the right hand side of the last equation, we see that the first criterion, namely summation of the Component VaRs to the total portfolio VaR, is indeed satisfied. By Taylor series expansion per the previous article, it can also be quickly verified that the other two criteria mentioned at the beginning of this article are also satisfied. Thus the term

$$\mathbf{p}_\mathrm{i}' \bullet VaRDelta(\mathbf{p})$$

is our choice for **Component VaR**.[3] It divides total VaR in an additive fashion, regardless of how the subset cashflow maps are selected. In other words, the cashflow maps can be partitioned by trades, by tenors, by assets, by instrument types, or by any other criterion, and our VaR-Delta-based definition of Component VaR remains valid.

Using Component VaR in Risk Management Reports

A trade-based Component VaR risk report might appear as in the following example:

Trade #	Trade Description	Mark-to-market	Component VaR	VaR-beta
547	Currency swap	$17,563	$14,845	0.61%
582	Currency option	$214,432	$141,861	5.82%
236	Bond option	$237,066	($121,490)	-4.99%
…	…	…	…	…
393	Knock-out option	$34,834	$15,765	0.65%
TOTALS		$3,452,764	$2,436,238	100.00%

Note that trade #236 has a negative Component VaR. This means that it serves as a "hedge" to the rest of the portfolio. If we deleted trade #236, the portfolio VaR would *rise*, by approximately $121,490. Similarly, trade #547 adds approximately $14,845 to portfolio VaR; were this trade deleted, portfolio VaR would fall by approximately this amount. When will the approximation be most nearly exact? This occurs when the Component VaR is "small." Thus, the Component VaR for trade #547 is likely to be the better approximation of how VaR will actually change if the corresponding trade is deleted. (Just as option deltas show option price sensitivity for small changes in the underlying price, VaRdelta – upon which Component VaR is based – is exact only for small changes in the cashflow map.)

The final column in the table above shows Component VaR as a percentage of total VaR, termed "VaR-beta." This number shows how portfolio risk is concentrated in the individual trades, as a fraction of overall risk. The terminology comes from the analogous nature of the definition of this quantity with that of beta's role in capital market theory, as explained in the last section of this paper.

Of course, partitioning the total portfolio by trades, per the foregoing, is not the only possibility. Similar component-risk reports might be analogously performed for various branch offices, for short-term vs. long-term cashflows, or for a host of other criteria.   The Component VaR report permits us to “slice and dice” total risk by any categories whatever, simply by taking a single inner product per category.

VaR-Beta

Dividing the Component VaR quantity by the total portfolio VaR yields the following calculation:

$$VaRBeta(\mathbf{p_i},\mathbf{p}) \equiv \frac{\mathbf{p_i'} \bullet VaRDelta(\mathbf{p})}{VaR(\mathbf{p})} = \frac{\mathbf{p_i'Qp}}{\mathbf{p'Qp}}$$

This ratio of quadratic forms is almost exactly analogous to the definition of beta in modern portfolio theory, and conforms to the same basic relationships. First, VaR-beta is additive, i.e.

$$VaRBeta(\mathbf{p_i}+\mathbf{p_j},\mathbf{p}) = VaRBeta(\mathbf{p_i},\mathbf{p}) + VaRBeta(\mathbf{p_j},\mathbf{p}),$$

which may be quickly ascertained from the previous formula. Second, the beta of the reference portfolio is unity, i.e.

$$VaRBeta(\mathbf{p},\mathbf{p}) = 1,$$

which is a restatement of the [Additivity] theorem.

Unlike its analogue in modern portfolio theory, however, it should be noted that there is here no absolute reference portfolio of cashflows (such as the "market portfolio"). Thus is important to observe that the VaR-beta is relative to the selection of the reference portfolio. This feature may be turned to some advantage, moreover, by providing *scope* to the VaR-beta concept. For example, a trade's cashflows might have a VaR-beta of 4% with respect to the portfolio of dealing desk from which it originated, a VaR-beta of 2% with respect to the branch office portfolio, and a VaR-beta of -1% with respect to the total institution's portfolio; all such numbers may be valuable in a risk management report.

Summary

The VaR-Delta concept has been shown to provide a reasonable basis for defining Component VaR. Via the [Additivity] theorem above, we find that total portfolio VaR may always be divided into additive Component VaRs. Moreover, these amounts are also meaningful, in the sense that they approximate the VaR actually lost (or gained) if the portfolio component were to be deleted (or doubled in size). An example Component VaR risk management report for trade components shows how the individual trades contribute to portfolio risk, and identifies those trades which act to hedge the remainder of the portfolio. Similar benefits are available via corresponding reports for any such categories which partition the total portfolio cashflow map. The VaR-beta metric of a set of component cashflows further clarifies the "percentage" of its contribution total risk, and serves as a useful addition to portfolio risk reports.

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[1] See "RiskMetrics™ -- Technical Document," Fourth Edition, December 18, 1996, J. P. Morgan/Reuters. Further publications and information are also available at the J. P. Morgan web site, URL http://www.jpmorgan.com/RiskManagement/RiskMetrics/RiskMetrics.html.

[2] This unusual result is analogous to the option-pricing theorem which states that the option value is equal to the sum of its value-denominated deltas, and apparently flows from the analogous homogeneity properties, namely the first-degree homogeneity of VaR(**p**) in **p**, in the same fashion as option formulas are homogeneous in the underlying asset prices.

[3] Note that the story is not entirely ended at this point. A "component" as used here means a "cashflow component," and this is not quite the same as a "portfolio component." To ensure that these are indeed the same, it may be required that the cashflow mapping function *m(.)* be **linear**. This means that for a suitably defined portfolio aggregation operator "+," $m(A+B)=m(A)+m(B)$ for all portfolios *A* and *B*. Both the mapping methodology itself and such institutional features as netting agreements may give rise to nonlinear mapping functions.

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